Row & Col. space

Last time:

Basis = "a linearly independent spanning set"

Dimen. = required # of linearly indep vectors required to make a basis

Thm/Note: The dimen of a space can only "be one" num. (So, if you make a basis of length k for V, then any other basis of V must also have length k)

Ex: Null space of $\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix}$

had a basis $\left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

So $\dim(\mathrm{Null}(\underline{A})) = 2$

Q: Same $\underline{A}$ ... can $\left\{ \begin{bmatrix} 5 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathrm{Null}(\underline{A})$? No, because too many vectors.

<u>Row space</u> $\overset{\text{DEF}}{=}$ span of row vectors of $\underline{A}$,

denoted as row$(\underline{A})$.

<u>Ex</u> $\underline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow \text{row}(\underline{A}) = \text{span}\{[1\ 2], [3, 4]\}$

$= \text{span}\{[1\ 0], [0\ 1]\}$

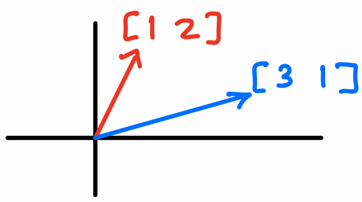
$\text{row}\left(\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\{[2\ 2], [0\ 0]\}$

$= \text{span}\{[2\ 2]\}$

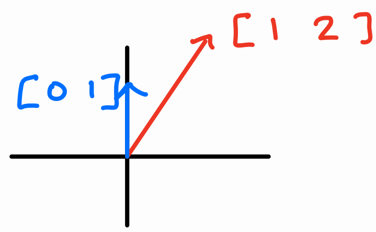
- Can "plot" row vectors in $\mathbb{R}^n$ just like col. vecs.
- Independence, basis, etc. all work the same for lin. comb. of row vectors.

☆ • <u>row</u> ops do not change <u>row</u> space of matrix

<u>Ex</u>: $\underset{\underline{A}}{\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \rightarrow \underset{\underline{B}}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \longrightarrow \cdots$



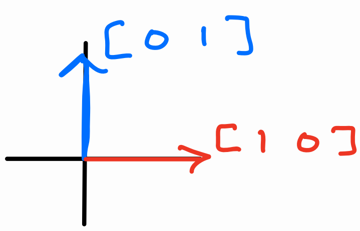
span$\{[1\ 2], [3\ 1]\}$ $= \mathbb{R}^2$



span$\{[0\ 1], [1\ 2]\}$ $= \mathbb{R}^2$

$\cdots \longrightarrow \underset{\underline{C}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

row$(\underline{A})$ = row$(\underline{B})$ = row$(\underline{C})$



$\underline{A}, \underline{B}, \underline{C}$ are "<u>row equivalent</u>"

span$\{[1\ 0], [0\ 1]\} = \mathbb{R}^2$

How to compute a basis for the row space of $\underline{A}$?

Ex: $\underline{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}$

1) Reduce $\underline{A}$ to echelon form* using row ops

$$\underline{A} \rightarrow \left(\begin{matrix}3 \text{ row} \\ \text{ops}\end{matrix}\right) \rightarrow \underbrace{\begin{bmatrix} \boxed{1} & -4 & -3 & -7 \\ 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{"}B\text{"}}$$

. .

(* non-reduced EF is fine)

2) locate pivot rows of $\underline{B}$ (row 1, row 2)

3) $\boxed{\text{Fact/Thm}}$ the pivot rows of $\underline{B}$ make a basis for row($\underline{A}$).

$$\text{row}(\underline{A}) = \text{row}(\underline{B}) = \text{span}\{\underbrace{[1\ -4\ -3\ -7],\ [0\ 1\ 1\ 3]}_{\text{basis for row}(\underline{A}).}\}$$

[The (row) rank of $\underline{A}$:

$$\text{rank}(\underline{A}) \overset{\text{DEF}}{=} \text{dimen}(\text{row}(\underline{A}))$$]

(rank($\underline{A}$) = 2 above)

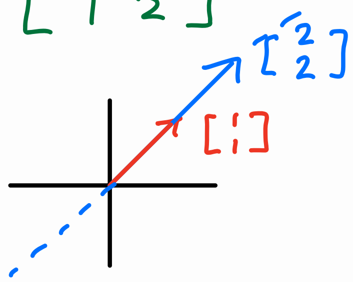
★ row rank = (# of pivot rows of (EF of $\underline{A}$))

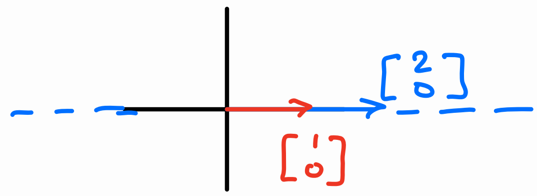
<u>Column space</u> $\mathrm{Col}(\underline{A})$ = span of columns of $\underline{A}$

<u>Ex</u>: $\mathrm{col}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$

<u>Note</u>: <u>row</u> ops modify column space:

$$\underset{\underline{A}}{\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}} \longrightarrow \underset{\underline{B}}{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}}$$





$\mathrm{col}(\underline{A}) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ $\qquad$ $\mathrm{col}(\underline{B}) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

How to find a basis for Col(A)

<u>Ex</u>: $\underline{A} = \begin{array}{c} \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \\ \begin{matrix} \underline{v_1} & \underline{v_2} & \underline{v_3} & \underline{v_4} \end{matrix} \end{array}$

$\mathrm{col}(\underline{A}) = \mathrm{span}\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}\}$, but ...

more cols than rows $\Rightarrow \{\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}\}$ is <u>lin dep</u>

1) Reduce $\underline{A}$ to echelon form using <u>row</u> ops

$$\underline{A} \to \dots \to \underbrace{\begin{bmatrix} \boxed{1} & -4 & -3 & -7 \\ 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\underline{B}}$$

2) locate pivot columns of $\underline{B}$ (col 1, col 2)

3) Use the corresponding columns of $\underline{A}$

NOT $\underline{B}$ as the basis for $Col(\underline{A})$

col 1, col 2 pivots, so use $\underline{v_1}$, $\underline{v_2}$ from $\underline{A}$

$$Col(\underline{A}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \right\}$$

column rank of $\underline{A}$:

$$\text{rank}(\underline{A}) = \dim(Col(\underline{A}))$$

(= # of pivot cols of (EF of $\underline{A}$)

= # of pivots " " ")

row rank and column rank are = # of pivots

in (EF of $\underline{A}$)

$\Longrightarrow$ row rank = col rank always !